ECON 897 Test (Week 3)
July 29, 2016

Important: This is a closed-book test. No books or lecture notes are permitted. You have **120** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. **(10 points)** State whether the following function is twice continuously differentiable, or not:

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

2. **(10 points)** Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ and $f'(x) \geq 1$ for all $x \neq 0$? If it does, show an example. If not, prove it.

3. **(30 points)**

   (a) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist and are finite, then $\lim_{x\to\infty} f'(x) = 0$.

   (b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f''(x)$ exist, then $\lim_{x\to\infty} f''(x) = 0$.

4. **(15 points)** State whether the following are linear subspaces, and prove your answer:

   (a) Let $W_n = \{f(x) \in P(F) | f(x) = 0$ or f(x) has degree exactly equal to $n > 1\}$. Is W_n a subspace of $P(F)$, where $P(F)$ is the space of polynomials?

   (b) Let $A = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = a_3 + 2\}$. Is A a subspace of $\mathbb{R}^3$?

5. **(15 points)** Consider P_2, the space of polynomials of degree 2. Let $a, b, c \in \mathbb{R}$, and define:

$$f_1(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)}, \quad f_2(x) = \frac{(x-c)(x-a)}{(b-c)(b-a)}, \quad f_3(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)}$$

   (a) Define what a basis of a subspace is.

   (b) Prove that $\mathbb{F} = \{f_1(x), f_2(x), f_3(x)\}$ is a basis for P_2.

   (c) Let $f \in P_2$ and write f as a linear combination of $f_1(x), f_2(x), f_3(x)$:

$$f(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x)$$

   What are the values of a_1, a_2 and a_3? That is, what are the "coordinates" of f with respect to $\mathbb{F}$?

6. **(20 points)** Let U and W be subspaces of a vector space V. Define:

$$U + W = \{u + w \quad | \quad u \in U, w \in W\}$$

Show that:

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W)$$

(Hint: use the Basis Extension Theorem).